Quantitative Reasoning – LAB 413

Inference for Proportions & Difference in Two Proportions

Week 11 Recap ▪ Lab 9

Instructor: **Subin Na**

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Today's Plan

1. **Quick Recap**
   - Bernoulli & Binomial Foundations
   - Sampling Distribution of $\hat{p}$
   - Confidence Interval for a Single Proportion
   - Hypothesis Test for a Single Proportion
   - Comparing Two Proportions
2. **RStudio Practice**

01 Bernoulli & Binomial Foundations

- **Bernoulli trial:** one trial $\to$ outcome 0 or 1 (failure/success).
- **Binomial distribution:** X = number of successes in n independent Bernoulli trials.

$$E[X] = np, \qquad \mathrm{Var}[X] = np(1-p)$$

- **Sample proportion** $\hat{p} = X/n \to$ point estimate for the population proportion p.

Example

Among $n = 200$ voters, $X = 120$ support Candidate A $\Rightarrow$ $\hat{p} = 120/200 = 0.60$.

02 Sampling Distribution of $\hat{p}$

Key Concepts

If the sample is random and large enough ($n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$), the sampling distribution of $\hat{p}$ is approximately Normal.

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- $\hat{p}$ is an **unbiased estimator**: on average $\hat{p} = p$.
- The **standard error** $\sqrt{p(1-p)/n}$ shrinks as n grows $\rightarrow$ more stable estimates.

03 Confidence Interval for a Single Proportion

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $z^* \approx 1.645$ (90%), 1.96 (95%), 2.576 (99%).

Plus-Four Adjustment

If counts are small, add 2 successes and 2 failures: $\tilde{p} = \dfrac{X+2}{n+4}$.

Conditions

- Random sample.
- At least 10 expected successes & failures.

Interpretation

- If we repeatedly sampled, $\approx C\%$ of intervals would capture the true p.

Example: Dogs vs. Cats — Setup

Scenario

A quick campus poll asks: "Do you prefer dogs over cats?"

- Random sample of $n = 600$ students.
- $x = 372$ answered "Yes."

Goal: estimate the true proportion p of all students who prefer dogs, and construct a **90% confidence interval** for p.

Example: Dogs vs. Cats — Solution

Given. $n = 600,\ x = 372,\ \hat{p} = 372/600 = 0.62.$
Conditions. $n\hat{p} = 372 > 10,\ n(1-\hat{p}) = 228 > 10.\ \checkmark$

$$SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.62(0.38)}{600}} = 0.0196$$

$$\hat{p} \pm z^* SE = 0.62 \pm 1.645(0.0196) = 0.62 \pm 0.032 \ \Rightarrow \ (0.588,\ 0.652)$$

The 90% CI for the proportion who prefer dogs over cats is $(58.8\%,\ 65.2\%)$, based on a method that would capture the true proportion about 90% of the time.

04 Hypothesis Test for a Single Proportion

Hypotheses

$$H_0 : p = p_0 \qquad H_a : p \neq p_0$$

Test statistic

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

Key Concepts

Reject H_0 when the test statistic is more extreme than the critical value — equivalently, when the p-value is smaller than α.

Example: Boba Tea vs. Coffee

Setup. Out of $n = 20$ students, 13 prefer boba tea over coffee, so $\hat{p} = 0.65$. Test whether this differs from 0.5.

$$z = \frac{0.65 - 0.5}{\sqrt{0.5(1 - 0.5)/20}} = 1.34$$

$$p\text{-value} = 2\,P(Z > 1.34) = 2 \times 0.0901 = 0.18$$

$p = 0.18 > \alpha = 0.05 \quad \Rightarrow \quad$ **fail to reject** H_0. Not enough evidence that the true proportion differs from 0.5. ("Fail to reject" does *not* prove H_0 true.)

05 Comparing Two Proportions

Goal: test or estimate the difference between two independent populations.

$$(\hat{p}_1 - \hat{p}_2) \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Conditions

- Two independent random samples.
- Each sample has ≥ 10 expected successes & failures.

Interpretation

- CI includes 0 $\rightarrow$ no significant difference.
- CI excludes 0 $\rightarrow$ significant difference.

Example: Pineapple on Pizza — Setup

Scenario

Among students, does support for pineapple on pizza differ by gender? Two independent samples:

- Women: 328 of 537 said "yes."
- Men: 234 of 532 said "yes."

Goal: construct a **95% confidence interval** for the difference $p_1 - p_2$ and decide whether the difference is statistically significant.

Example: Pineapple on Pizza — Solution

Given. $\hat{p}_1 = 0.611$ (women), $\hat{p}_2 = 0.440$ (men). Both samples meet the ≥ 10 successes & failures condition. ✓

$$SE = \sqrt{\frac{0.611(1-0.611)}{537} + \frac{0.440(1-0.440)}{532}} = 0.030$$

$$(\hat{p}_1 - \hat{p}_2) \pm z^* SE = (0.611 - 0.440) \pm 1.96(0.030) = 0.171 \pm 0.059$$

$$\Rightarrow \; 95\% \text{ CI} = (0.112,\ 0.230)$$

The 95% CI for the difference (women − men) is (0.112, 0.230). Since it excludes 0, women are significantly more likely to approve pineapple on pizza.

Key Concepts

- When n is large enough, the sampling distribution of $\hat{p}$ is approximately Normal $\rightarrow$ we can use z procedures.
- A confidence interval is always **estimate** $\pm$ **margin of error**: for proportions, $\hat{p} \pm z^* \times SE$.
- A hypothesis test compares the sample statistic to the null value using a z-score (distance in standard-error units).
- Always check conditions first: random sample and large-sample thresholds (≥ 10 expected successes & failures).
- Two-proportion procedures follow the same logic as two-sample mean tests — only the formulas change.

RStudio Practice

Comparing Means: Two-Sample t-test (McGrath Experiment)

See `Lab9.Rmd` / `Lab9_1105.R`